The Five Mysteries of Dynamic Systems

A Concept Map for Undergraduate Students

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INTRODUCTION

As a student of engineering in the 1970s with a particular interest in control engineering I developed some expertise at that time in the use of the Laplace Transform to describe and analyse dynamic systems. In spite of this expertise and the increasing familiarity which resulted from using the method for research and teaching, there has remained an uncertainty in my understanding which can best be illustrated by the question: "What IS '*s*'?" This is a question I distinctly recall asking myself and others (perhaps even lecturers) at the time. Every engineering student, including me, has quickly learned to answer that question with the fact , where is an exponential decay rate and is a sinusoidal frequency. But decay of what?, and frequency of what? And anyway what does '*s*' signify as the independent variable of the function , the Laplace transform of ?

The origin of the motivation for this paper is in another similar question which I posed to my colleagues at morning coffee break a number of years ago. The question was/is "Why ' ' ?" The purpose of this paper is to review these questions in relation to and , the 'independent' variable of the Laplace, Fourier and z-Transforms. In my review of this question I have suggested five quantities which have been a source of some 'mystery' to students of all levels including third level engineering students. These five mysteries are listed as

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- , the ‘imaginary’ number
- , the constant known as e
- , negative frequency
- , complex frequency
- , complex frequency of discrete-time signals

I hope to bring these five quantities together under one roof so to speak in such a way that students are more comfortable thinking about them and wondering what they mean as they use them in the usual way to solve engineering problems. So the purpose of this paper is to identify the individual mysterious quantities, discuss them and then most importantly relate them to each other. This paper will not try to solve the mysteries!

These five quantities are the basic components of the tools which are used to describe linear dynamic systems, namely the Fourier, Laplace and z Transforms. [1], [2], [3]. I have developed three diagrams which I have called concept diagrams. These diagrams provide an overview of how the quantities combine to form the basis for the transform tools. It is hoped that these diagrams will serve as a convenient memory map of the subject, will shed some light on the mysterious aspects of the subject and will stimulate debate among both students, lecturers and practitioners.

1 BRIEF REVIEW OF THE MYSTERIES.

Each of these quantities is more or less mysterious according to the different stages in students education. Each has its own question or mystery..

1.1 The ‘imaginary’ number ,

The imaginary number, , (for electrical engineers to distinguish it from the symbol used for current), at the heart of the subject, is first met at second level and is of course mysterious because it is an ‘imaginary’ number. The fact that it is used to facilitate the solution to quadratic equations is not that helpful to student’s insight into the mystery. Its use in electrical engineering to mathematically describe the electrical quantities of *reactance* and *voltage* and *current pha*sors makes it more familiar but perhaps no less mysterious. Its subsequent appearance in the ‘s’ and ‘z’ planes I think further deepens the mystery.

1.2 The constant known as

Why *e* = 2.71828...? Why this particular number? There are many answers to this question. One answer is the fact that It is the number which in the function

. This fact is central to the solution of linear constant coefficient differential equations. The number, 2.71828...? , not a rational number, is conveniently referred to as a symbol, the letter . The importance of the function and its derivatives is due to the fact that weighted sums of are solutions to linear ordinary differential equations. A second answer is that is the number for which , Eulers Great Equation (in itself a wonderful mystery, but not the subject of this paper)

1.3 Frequency, and Negative Frequency,

The concept of frequency, the rate at which things repeat, is not obviously mysterious. We as humans have become used to the concept of frequency and giving it a numerical value. One aspect of its mystery is in the importance of (radians/second) as opposed to (cycles/repitітions/second) when it comes to the mathematical representations. A second and very interesting aspect of the mystery of lies in the frequent occurrence in the maths of this area of – , negative frequency. Negative numbers are actually mysterious. They have become less so due to familiarity of use from an early age and due to their usefulness. Negative frequency? How does something vibrate negatively? It either vibrates or it doesn't. Well this is mysterious until you see that it can be used as a negative number in functions like . Then simply becomes a negative number. We will see that is a vector which rotates in one direction while rotates in the opposite direction.

A second aspect of the mystery of and – is to do with the mathematical representation of sinusoidal signals. Sinusoidal signals are the basis for Fourier analysis of signals. The most convenient method of representing sinusoids is the complex exponential, . This is described in the first concept diagram below.

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1.4 Complex Frequency, and

The mysteries of and are considered together since they are both manifestations of the same quantity, *complex frequency*. is for continuous time signals and is for discrete time signals.

' ' is the symbol used to represent the independent variable of the Laplace transform. The fact that *s* is a complex variable makes it difficult and mysterious for students and others. The fact that it represents *frequency* is mysterious. The term, *complex frequency*, is the cause of some of the mystery. How can frequency be complex?

Then, how does fit into this picture. We know that the imaginary axis (frequency) of the -plane folds round onto the unit circle in the -plane. But what does this mean?

This paper attempts to provide a useful starting point for students and others to answer these questions and the other questions suggested above by bringing together in a graphical and diagrammatic form the main components of the topics.

2 THE CONCEPT DIAGRAMS

2.1 Introduction

Three concept diagrams are presented. The first, shown in Figure 1 brings the five mysteries together to form the mathematical entity which is used to describe linear, continuous-time and discrete interval (time) signals and systems. The second diagram, shown in Figure 2, describes how the two complex frequency 'objects', ' ' and ' ' are presented as 2-dimensional *frequency domain* independent variables, and further describes how those independent variables relate to their *basis* functions, and and to the frequency domain graphical representation of these functions. Finally the third diagram presents an overview of the interconnections and relationships between all the main quantities involved in linear dynamic signal and system analysis.

2.2 Concept Diagram 1. The Basis Functions, The complex exponentials.

The first diagram shows how the five mysteries taken in sequence build the fundamental time functions used in Fourier, Laplace and z transform analysis. These functions, known as *basis functions*, are various forms of the 'time' dependant complex exponential. The term basis function means a function, combinations of which can be used to develop more complex functions. Time refers to continuous time, and discrete time interval, .

The first graph in concept diagram 1 shows how is used to represent a two dimensional quantity, as a single entity, the complex number .

The second graph shows how this same quantity can be represented using a complex exponential through the 'magic' of Euler's marvellous equation. i.e. . The two components of are now the length of , ( ) and the angle of , ( ).

The next two graphs shows what happens when the angle part of is caused to vary at a certain *rate*, such that , and . In one case varies continuously with time and in the other case varies in discrete steps with . The result in both cases is a spiral with constant amplitude as shown. The spiralling phasor is shown in three representative positions as it moves along the time axis, spiralling as it goes.

The final two graphs show what happens to the 'spiralling' phasors when the complex exponential is modified by multiplying it by a real exponential, causing the amplitude to decay (or grow) with time.

2.3 Concept Diagram 2. The relationship between the independent variables and and their functions, and .

This diagram focuses on the graphical representation of the basis functions, the complex exponentials, to shed light on the relationships between the time functions and their representations in their respective frequency domains.

Figures 2(a) and 2(b) show the discrete time and continuous time functions respectively. Both are similar. They are created in MATLAB starting with the continuous time frequency . The transformation was used to produce a corresponding discrete time frequency . Different aspects of the functions are shown in each diagram. In 2(b) the spiral is decomposed into the real and imaginary components and they are seen to be cosine and sine waveforms respectively. In 2(a) the decay envelope is shown with the spiral.

It is worth emphasising that the equations for the time functions are essentially the same even though they are normally used in different forms. They are given in the figure and are repeated here:

Continuous time:

Discrete time:

2.4 Concept Diagram 3. The Complete Picture

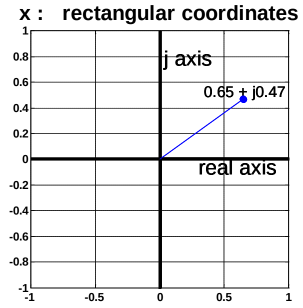
The third diagram, Figure 3 below, is presented to complete the picture. It shows the independent variables, as the starting point. Built on top of each independent variable is its function, . The transform relationships between the functions are then examined. Finally the relationships between the independent variables are shown.

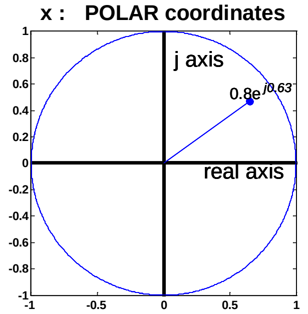
This is not a new diagram. Any teacher of signals and systems has thought about or drawn some version of this diagram. Versions are available in many signals and systems text books. [1], [2], [3]. I present this version of the diagram for consideration and for debate. In fact each diagram is presented hopefully as a source of debate about the beautiful mysteries of the mathematical description of dynamic systems.

Concept Diagram 1: The Basis Functions

Shows how the five mysteries are combined together through Euler's Great Equation to form the *basis functions* for Fourier , Laplace and z-transform analysis.

Euler's Great Equation:





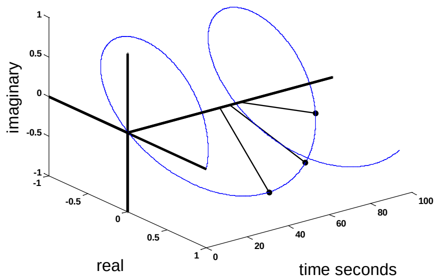
Euler's equation allows the rectangular coordinates to be represented in terms of polar coordinates.

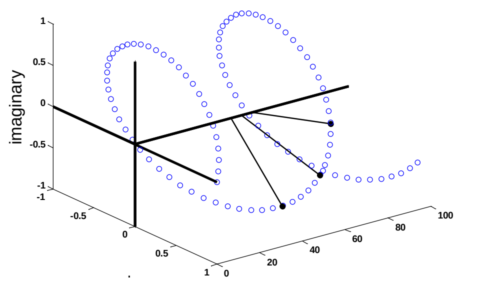
When the complex number is given in polar form it is easily seen to be a vector with magnitude, and angle, . It is now easy to see how this vector can be made to rotate at a constant rate by letting the angle vary at a constant rate,

radians/second, i.e.

Allow to vary continuously with time:

Allow to vary in discrete steps, n at frequency :

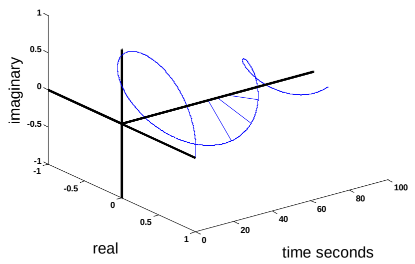




Shown above are the Fourier basis functions for continuous and discrete time. They start at and go on forever with the same amplitude. These functions can be modified to account for signals which start at time =0 and decay or grow in amplitude. The complex exponentials are multiplied by **real** exponentials.

-
- represents *complex frequency*, , where is the decay rate and is the oscillation rate.

-
- represents *complex frequency*, , where is the decay rate and is the os



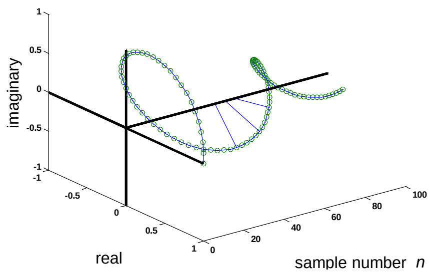
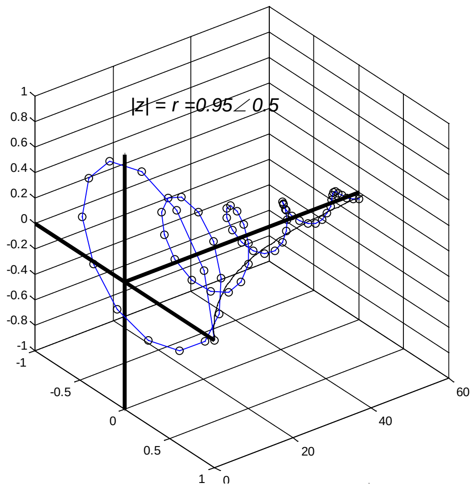
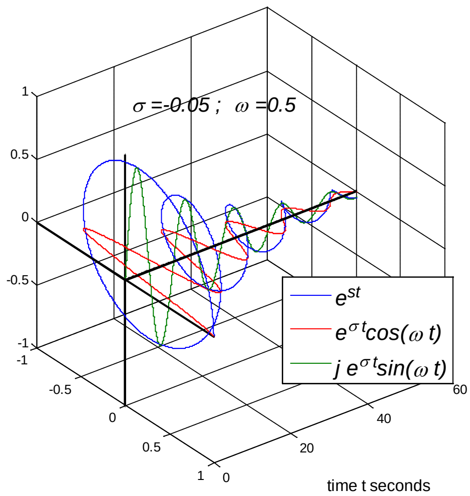


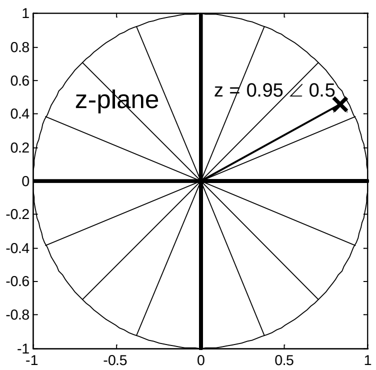
Figure 1. Concept Diagram 1: The Basis Functions

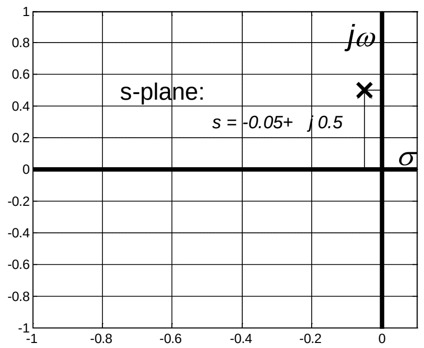




2(a)

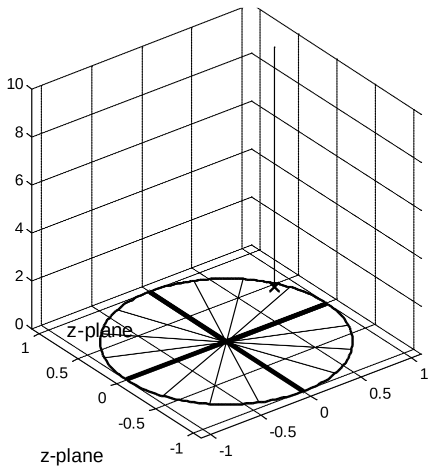
2(b).

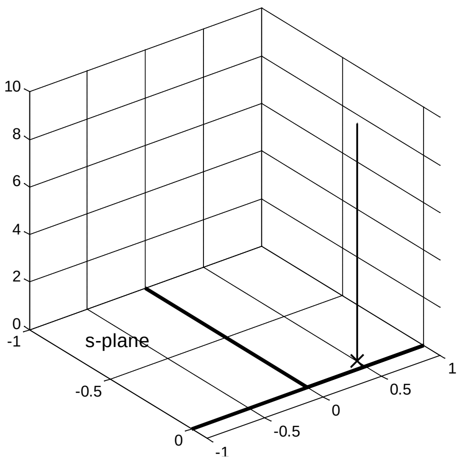




2(c) x marks the frequency, of . . The frequency is 0.5 rad/sample and the decay rate is .

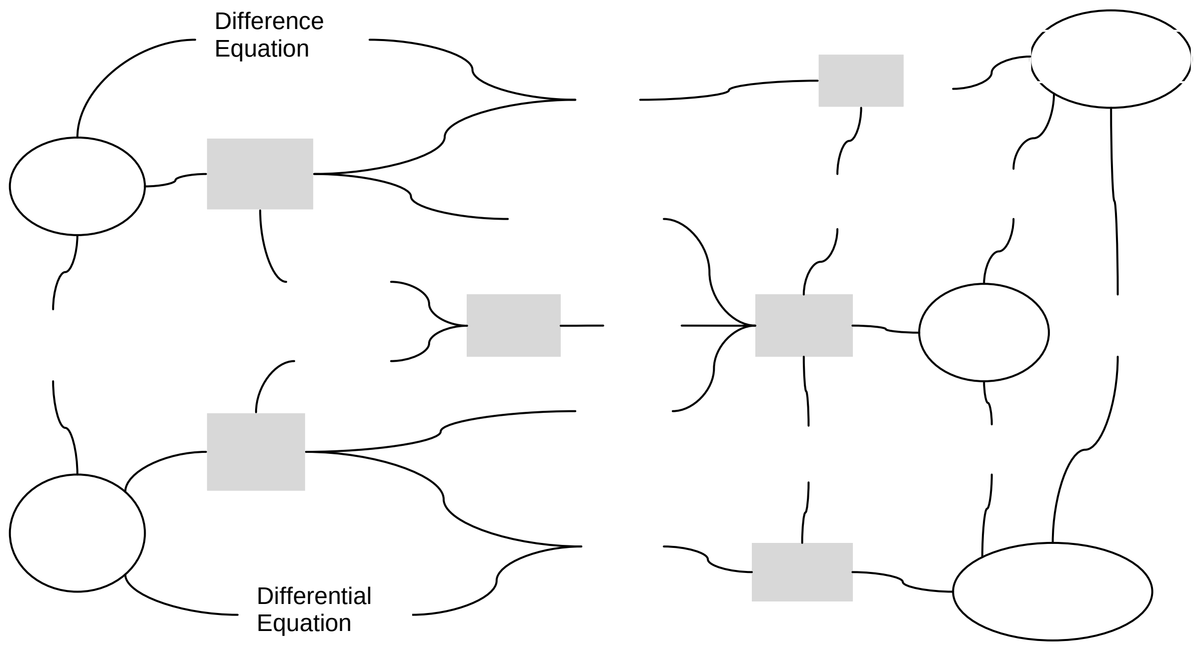
2(d) x marks the position in the s-plane corresponding to the frequency of in figure 2(b). ω is 0.5 rad/s and σ is -0.05. This is a decay rate.





2(e). These plots show exactly the same information as those in 2(c) and 2(d). They use 3-D to show the presence of the transformed functions, and as delta functions. X marks the frequency in each *plane* and the vertical line shows the delta function.

Figure 2. Concept Diagram 2: The relationship between the independent variables and and their functions, and .



Legend:

	Independent variables:
	Functions:
	Transformation of the independent variables: . These convert from one form of frequency domain to another
	Domain Transforms : Laplace, Z, Fourier, etc:

Figure 3. Concept Diagram 3. The complete picture. Overview of interrelationships between the different representations of linear signal, linear systems and their transforms.

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