# What makes a counterexample exemplary?

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## INTRODUCTION

Conventional calculus education has emphasized "good" functions and examples; thus, students are accustomed to using familiar skills and manipulating signs rather than focusing on reasoning and verifying mathematical concepts, definitions, and theorem conditions. This type of teaching and learning orientation generates misconceptions among students and may be explained by the generic extension principle proposed by Tall [1]. According to Skemp, a concept, for its formation, requires a number of experiences – or examples – that have something in common [2]. Generating examples of mathematical objects may be a complicated task for both students and teachers [3]; however such tasks yield substantial educational potential. In mathematics education, requesting that students generate examples is a particularly valuable tool [4]. Lakatos argued that the more students practice in creating their own counterexamples the more likely they see them as infinite classes of examples rather than isolated and irrelevant pathological cases [5]. Furthermore, in advanced mathematical thinking, the role of counterexamples as an integral part of problem solving strategies and as a way of mathematical thinking.

There are at least two important roles that examples play in mathematics education. One is of interest to teachers and designers of instructional materials, while the other is of interest to researchers [6]. Peled and Zazlavsky classified counterexamples into *specific, semi-general* and *general* depending on the extent to which expert mathematicians provide an insight how to construct similar counterexamples or generate an entire counter-example space [7]. Zazlavsky and Ron indicated that "students' understanding of the role of counterexamples is influenced by their overall experiences with examples." [8] Zazkis and Chernoff also suggested that "the convincing power of counterexamples depends on the extent to which they are in accord with individuals' example spaces." [9] This study, based on the perspective that "examples as a research tool that provides a 'window' into a learner's mind" [10]. Data were collected from engineering students to explicate these studies and discuss variations within the classifications.

## METHODOLOGY

The participants comprised 112 first-year engineering students at a university of technology in Taiwan who previously completed courses of derivative and definite integrals. The questionnaire contained three false mathematical statements, and was designed to assess student ability to generate counterexamples regarding differentiation and integration. These mathematical statements required an understanding of and the ability to generate basic differentiation and integration counterexamples. These statements enabled the students' performance regarding the example generation to be analysed.

Statement 1: If $\int_a^b f(x)dx \ge \int_a^b g(x)dx$ then $f(x) \ge g(x)$, $\forall x \in [a,b]$.

Statement 2:If $f(x)$ and $g(x)$ are all differentiable and $f(x) \succ g(x)$, $\forall x \in (a,b)$ then $f'(x) \succ g'(x)$, $\forall x \in (a,b)$.

Statement3:If $f(x)$ and $g(x)$ are all differentiable and $f'(x) \succ g'(x)$ , $\forall x \in (a,b)$ then $f(x) \succ g(x)$, $\forall x \in (a,b)$ .

The primary data sources were the written responses to the questionnaire and interviews, which was administered to the students in their calculus classes after they completed the course of differentiation and integration. The students were asked to determine the validity of the mathematical statements and justify their answers. The results of the questionnaire necessitated further investigation into the example generation of students. Semi-structured interviews were carried out with 15 engineering students. The interviews were video- and audio-taped and time was allotted for each task so that interviewees had enough time to answer all the questions. During each interview and were asked to think aloud while they were solving the tasks so that we could describe their responses and strategies as well as make inferences about their examples and counterexamples.

The counterexamples that the participants generated were gathered as data. To facilitate characterizing student performance and assessing how the participants responded to the false statements, the categories and those listed in Table 1. The analysis focused on identifying the students' counterexamples used to create meanings for the statements and the justifications provided.

## RESULTS

Table 1 shows the distribution of the response types of engineering students, and the results show that the response types are related to mathematical statements. For example, most students asserted that Statement 1 was correct and few students generated correct counterexamples. A substantial number of students generated counterexamples for the differentiation statements, and more students generated counterexamples for Statement 2 compared with those for Statement 3. This finding indicated that at least 41% of students have failed to identify that the three mathematical statements were false. This may be because students lack conceptual understanding and rely on their intuition, resulting in overgeneralization.

*Table 1.* Types of Responses to the three false mathematical statements

| The statements | S1 | S2 | S3 |
|---|---|---|---|
| Types of Responses(N=112) | N(%) | N(%) | N(%) |
| Asserting that the statement is correct | 73(65.2%) | 49(43.7%) | 46(41.1%) |
| Left blank, no relevant knowledge | 10(8.9%) | 12(10.7%) | 17(15.2%) |
| Used positive examples as demonstrations | 2(1.8%) | 10(8.9%) | 5 (4.4%) |
| Used examples as demonstrations but did not support it or making logical errors | 61(54.5%) | 27(24.1%) | 24(21.5%) |
| Asserting that the statement is incorrect | 39(34.8%) | 63(56.3%) | 66(58.9%) |
| Left blank, no relevant knowledge | 8(7.1%) | 2(1.8%) | 11(9.8%) |
| Narrated a statement that was false instead of providing a counterexample to refute it | 7(6.3%) | 6(5.4%) | 10(8.9%) |
| Provided a counterexample that failed to refute a false statement | 16(14.3%) | 18(16.1%) | 26(23.2%) |
| Correct use of counterexample | 8(7.1%) | 37(33%) | 19(17%) |

Statement1: If $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ then $f(x) \geq g(x)$, $\forall x \in [a,b]$.

Overall, 65% of the participants asserted that Statement 1 was correct and generated examples for verification; however, most students generated converse statement examples (e.g., Selina):

> Selina: Greater integrals yield greater functions. Thus, this statement is correct. For example, $f(x) = x^2 + 1$ and $g(x) = x^2$, f(x) is larger than g(x), with an integral from 0 to 1. The integral of f(x) is 4/3 is also greater than the integral of g(x) is 1/3.

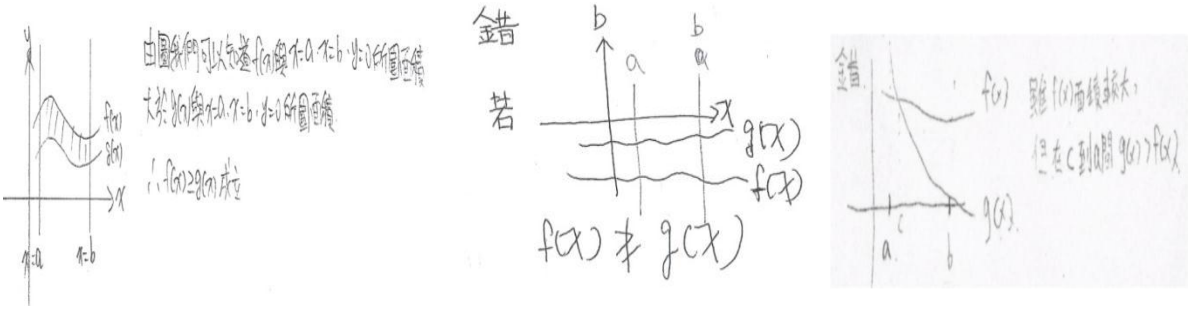
Approximately 21% of students used graphical representations to generate examples or counterexamples, including Keven and John. Although they connected integrals and areas, they either made misjudgements about or do not understand the true relations between integrals and areas. For instance, John believed that the integral was not related to the area above or below x-axis.

Keven: This statement is correct. The integral represents the area. With a greater integral, the area is greater [Fig. 1(a)] and this is correct. Here, a greater integral represents that the graph is located higher and thus the function is greater.

John: I think the statement is false. Greater integral does not imply greater function. Integral is the area. Thus, a greater integral means a greater area. Take a look of what I have drawn [Fig. 1(b)], the area surrounded by f(x), x = a, x = b and x-axis is larger than that of g(x). However, f(x) is smaller than g(x).

George was one of the few students who successfully generated counterexamples, intuitively judging the statement as false; however, he could not generate a counterexample until he noticed '$\forall x \in [a,b]$'.

George: The key point is that for all x in the interval [a,b]. In this graph [Fig. 1 (c)] I drew, the area surrounded by f(x), x = a, x = b and x-axis is larger than that of g(x); therefore, the integral of f in [a,b] is greater than the integral of g in [a, b]. But, the function value of f in the interval [a, c] is smaller than the function value of g in the interval [a, c]. Therefore, the statement is incorrect.




(a) (b) (c)

Fig.1. Example and Counterexamples of Statement 1

Statement2: If $f(x)$ and $g(x)$ are all differentiable and $f(x) \succ g(x)$, $\forall x \in (a,b)$ then $f'(x) \succ g'(x)$, $\forall x \in (a,b)$.

For this statement, 40.2% of the students generated examples or counterexamples for verifying or refuting the statement. Most students (e.g., Tanya and Dan) failed to notice the interval and thus failed to generate correct examples or counterexamples.

Tanya: This should be correct. Given $f(x) = 3x^2 + 5x + 2$ and $g(x) = x^2 + 6x + 1$, then $f'(x) = 6x + 5$ and $g'(x) = 2x + 6$. When x equal to 1, f of 1 is10,and g of 1 is 8, and $f'(1)$ is 11, and $g'(1)$ is 8. Thus, this statement is correct.

Dan: If $f(x) = x^2 + 1$ and $g(x) = x$, f(x) is greater than g(x), and $f'(x) = 2x$, $g'(x) = 1$ and $f'(x)$ is greater than $g'(x)$. Thus, this statement is correct.

Cindy noticed the effects of constants in the differentiation process.

Cindy: This statement is false, because constant terms become 0 after differentiation and constant terms influence the size of the function. For instance, f(x)=3x+1, g(x)=3x-5, f(x) is greater than g(x), but $f'(x)$ equal to $g'(x)$.

Few students (e.g., Tom and Lisa) used graphical representation or the slope of tangent to argue that the mathematical statement was false (Fig. 2).

> Tom: I think a greater function does not imply a greater derivative. This statement is false. Thus, I would like to find an example to prove that larger functions imply higher graphs. The graph of f(x) is above that of g(x). The g(x) graph is a straight line with a constant slope. However, the slope of f(x) in the interval of (a,b) is not always greater than that of g(x). Therefore, I proved that this statement is false.

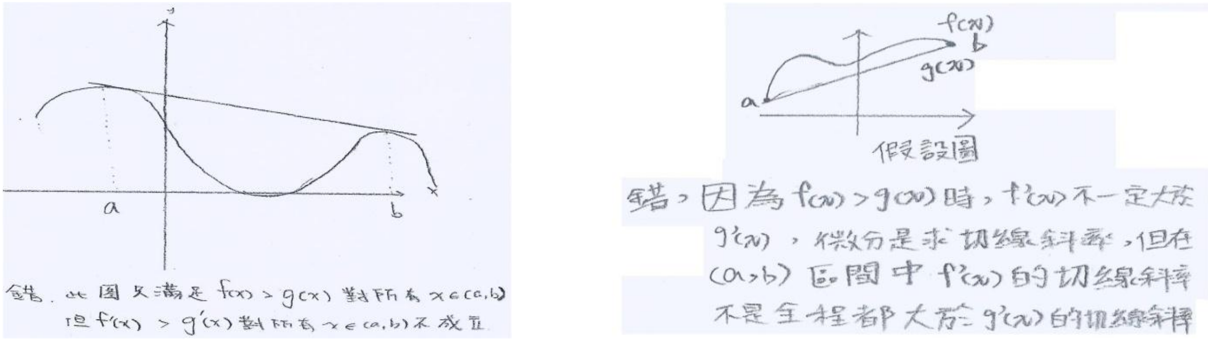
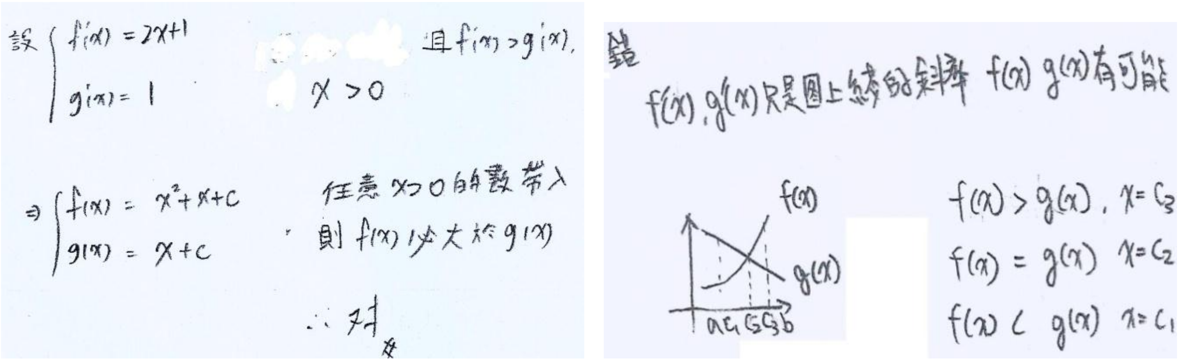



Fig. 2. Lisa's and Tom's Counter-examples of Statement 2

Statement3: If $f(x)$ and $g(x)$ are all differentiable and $f'(x) \succ g'(x)$ , $\forall x \in (a,b)$ then $f(x) \succ g(x), \forall x \in (a,b)$ .

Statement 3 is a converse statement of Statement 2, however, the students demonstrated a distinct performance in response to these statements. Only 17% of students successfully generated counterexamples for both Statements 2 and 3. By contrast, 38.3% of students failed to generate examples or counterexamples, indicating that Statement 3 was more challenging compared with Statement 2. However, 44.7% of the students neither noticed the interval nor generated correct examples or counterexamples when verifying or refuting the statement. For instance, Johnson assumed that $f'(x) = 4x$ , $g'(x) = 2x$ , $f'(x)$ is greater than $g'(x)$ , and $f(x) = 2x^2$ is greater than $g(x) = x^2$ . He did not realize that this example did not satisfy the condition of the statement, and ignored the range of x. In contrast to Johnson, Abbie also said that the statement was true, indicating that x was positive [Fig. 3(a)] and used algebraic representation to generate an example. But she neglected the meaning of the arbitrary constant c. She assumed that $f'(x) = 2x + 1, g'(x) = 1, f'(x)$ is greater than $g'(x)$, for all $x \succ 0$, and $f(x)$ is $x^2 + x + c, g(x)$ is $x + c$ , $f(x)$ is greater than $g(x)$ , for all $x \succ 0$ . The 19 students who successfully generated counterexamples all noticed the effect of '$\forall x \in$(a,b)' in the mathematical statement. For instance, Anna connected derivatives and the slope of the tangent, using graphical representation to generate counterexample [Fig. 3(b)]; she also indicated that f(x) is not greater than g(x), for all x in the interval (a,b).




(a) (b)

Fig. 3. Counterexamples of Statement 3

## CONCLUSION

This study explored how first-year engineering students performed when making inferences about false mathematical statements. The results showed that compared with integral concepts, the students were more likely to generate counterexamples for derivatives. This finding supports the argument that students ability to generate counterexamples is highly related to the context [11]. The graphical representations connected with integrals and derivatives are the area and the slope of the tangent, respectively. However, the area representation is more complex compared with the slope of tangent. This likely explains why few students generated counterexamples in response to Statement 1. Regarding Statement 1, although using graphical representation is a simple method of generating counterexamples, the students preferred using algebraic representation. Thus, visual-image knowledge and simple drawing skills related to functions are crucial for students to generate appropriate counterexamples. Finally, the differences in the performance of students when processing the three statements showed that the demonstration of flexibility and reversibility of thinking is crucial to completely understanding differentiation and integration in calculus [12].

To decide whether a mathematical statement is correct or incorrect, students must thoroughly understand the concept and logical reasoning. In this study, the engineering students demonstrated insufficient understanding of differentiation and integral concepts; therefore, they could not generate proper counterexamples to refute the false mathematical statements. They often fail to check that the conditions are satisfied. We find that this is because they are not fluent in using appropriate terms, notations, properties, or do not recognize the role of such conditions. To improve the understanding of engineering students regarding calculus concepts, teachers must understand students misconceptions [13] and spend additional time interpreting relevant concepts. Counterexamples can be used for assessment of learners' understanding in a broad sense. Furthermore, uses of counterexamples for teaching, in selecting instructional examples it is important to take into account learners' preconceptions and misconceptions. The counterexamples generated by participants (e.g., George, Tom, Anna) could serve as pivotal or bridging examples [10]. In our opinion, the power of these examples is also testified by the fact that they tend to impress in the memory of students, because of their peculiar features, thus further helping their learning process.

Because numerous students could not distinguish which examples satisfied or did not satisfy the counterexample conditions, they failed to generate accurate counterexamples. For instance, the students could not comprehend the sufficient and necessary conditions in logical statements, particularly the 'if-then' statements used in calculus concepts. Certain students used the counterexamples for converse statements as the counterexample to refute statements, whereas others used examples that satisfied the statements as counterexamples. These findings are similar to those of Zaslavsky and Peled, who indicated that student-teachers and mathematics teachers used incorrect counterexamples for given statements [3]. In addition, most students neglected the sufficient condition '$\forall x \in (a,b)$', which was the key to successfully generating counterexamples or examples. Therefore, mathematics education must enable students to understand the sufficient and necessary conditions in logical statements and familiarize them with using logical language, particularly the 'if-then' statement in calculus. Teaching strategies that require generating counterexamples may provide a viable teaching orientation.

**References**

[1] Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 3-21). Kluwer: Dordrecht.

[2] Skemp, R. (1987, second edition) *The psychology of learning mathematics*, London, UK, Penguin Books.

[3] Zaslavsky, O. & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student-teachers: The case of binary operation. *Journal for Research in Mathematics Education*. 27(1), 67–78.

[4] Watson, A. & Mason, J. (2005). Mathematics as a Constructive Activity: The Role of Learner generated examples. Erlbaum, Mahwah, NJ.

[5] Lakatos, I. (1976). Proofs and Refutations: The Logic of Mathematical Discovery. Cambridge University Press: Cambridge.

[6] Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th International Conference on the Psychology of Mathematics Education* (Vol. 1, pp. 126-154). Prague: IGPME.

[7] Peled, I. & Zaslavsky, O. (1997). Counter-examples that (only) prove and counter-examples that (also) explain. Focus on Learning Problems in Mathematics, 19(3), 49–61.

[8] Perkins, D.N. & Salomon. G. (1989). Are cognitive skills content bound? Educational Researcher, 18(1), 16-25.

[9] Zaslavsky, O. & Ron, G. (1998). Students' understanding of the role of counter-examples. In Olivier A. & Newstead K. (Eds.), *Proceedings of the Twenty-second Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 225–232). Stellenbosch, South Africa: IGPME.

[10] Zazkis, R., & Chernoff, E.J. (2008). What makes a counterexample exemplary? *Educational Studies in Mathematics*, *68,* 195-208.

[11] Zazkis, R., & Leikin, R. (2007). Generating examples: From pedagogical tool to a research tool. For the Learning of Mathematics, 27, 15–21.

[12] Perkins, D. N. & Salomon,G. (1989). Are Cognitive Skills Context-Bound? *Educational Researcher*, Vol. 18, No. 1, 16-25.

[13] Haciomeroglu, E. S., Aspinwall, L. & Presmeg, N. (2009). The role of reversibility in the learning of the calculus derivative and antiderivative graphs. In S. L. Swars., D. W. Stinson & S. Lemons-Smith (Eds.), *Proceedings of the 31th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*(Vol.5, pp.81-88). Atlanta, GA: Georgia State University: IGPME.

[1] Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*(4), 487–500.