# Engineering Students' Difficulties in Learning Complex Numbers

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## INTRODUCTION

The 21st century has seen rapid technological development with a steady shift away from manual and low-skills jobs towards those that require higher-levels of problem-solving skills, many of which are mathematical in nature. To face the challenges of the 21st century, engineers would be better served if they possess a strong grasp of the fundamental mathematics specific to the field and have the ability to apply theory to practice [1]. As one of the fundamental competencies in engineering education, electrical and electronic engineering (EEE) students should be able to explain the lifecycle of products, processes, and systems that range from simple to incredible complex of electrical circuit systems for both direct and alternating currents [2]. In the case of alternating current systems (AC), EEE engineers could abstract the concepts and uses of complex numbers to explain the characteristic of the inputs, outputs, and the system arrangements of the AC systems. Hence, possessing a strong grasp on the concepts, uses, and abstractions of complex numbers is very important for EEE engineers. However, the concepts of complex numbers are very challenging. There are three hierarchically ordered abstract forms, i.e. basic, polar, and exponential forms. Moreover, each form has two representations, i.e. algebraic and geometric representations. On the other hand the uses of complex numbers require some conceptual knowledge necessary to solve polynomials, e.g. de Moivré's Theorem. Furthermore, the abstractions of complex numbers require some knowledge to solve AC problems, e.g. Ohm's Law, phasors, and impedances. Hence, learning complex numbers could be very challenging for some EEE students. In this paper I reported an in-depth analysis including analysis on the complex numbers test, students' written works, and lecturers' responses on interview to answer the following question: *What kinds of difficulties faced by EEE students in grasping the concepts, uses, and abstractions of complex numbers?*

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# 1 THE CONCEPTS, USES, AND ABSTRACTIONS OF COMPLEX NUMBERS

In this section the three hierarchically ordered forms of complex numbers were described. The descriptions for each form include its two representations, i.e. algebraic and geometric representations. Followed by descriptions about how complex numbers can be used to solve polynomials problems. At last, the descriptions on electrical circuits and the abstractions of complex numbers in AC systems. These abstractions include the concepts of phasors and impedances.

## 1.1 The three hierarchically ordered forms of complex numbers

There are three hierarchically forms of complex numbers should be strongly grasped by EEE students, i.e. basic, polar, and exponential form. The *basic form* of complex numbers can be represented algebraically as $z = x + jy$, where $x$ is the real part, $y$ is the imaginary part, and both $x, y \in \mathbb{R}$, and geometrically as ordered pairs of $(x, y)$ in Cartesian coordinates in Argand diagram. Whereas the *polar form* can be represented algebraically as $z = r(\cos\theta + j\sin\theta) = r\angle\theta$ where $r$ is called the *modulus* of $z$, i.e. $r = |z| = \sqrt{x^2 + y^2}$ and $\theta$ is called the *argument* of $z$ and denoted as $\arg(z)$. It is an angle such that $\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$. The *principal value* of $\theta$ is taken in the interval $]-\pi, \pi]$. Geometrically as ordered pairs of $[r, \theta]$ in polar coordinates in Argand diagram. To avoid any confusion, the square brackets for polar coordinates are used to distinguish between round brackets for Cartesian coordinates. Finally, using the Euler's formula, it is allowed to write:

$$\cos\theta + j\sin\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = e^{j\theta}.$$

Hence, the complex numbers can be represented algebraically as $z = re^{j\theta}$. This representation is called the *exponential form* of complex numbers. This form has the same geometric representation as in polar form, i.e. $[r, \theta]$. Table 1 presented the summary of the three forms.

*Table 1*. The three hierarchically forms of complex numbers which taught for year one EEE students in their engineering mathematics course

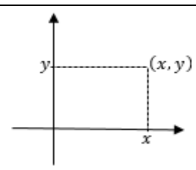
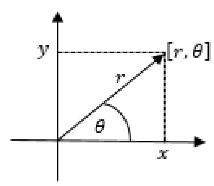
| Concepts of complex numbers | Algebraically | Geometrically |
|---|---|---|
| Basic form | $z = x + jy$ |   |
| Polar form | $z = r(\cos\theta + j\sin\theta) = r\angle\theta$ |   |
| Exponential form | $z = re^{j\theta}$ | |

Table 1 shows how these three hierarchically ordered forms of complex numbers different in the nature both algebraically and geometrically. Hence grasping the exponential form require a strong grasp on the first two forms. Again, algebraically and geometrically.

### 1.2 The *n*th roots of a complex number $a$

This subsection described the application of de Moivré's Theorem to produce the solutions of $z^n = a$, for a complex number $a$. De Moivré's Theorem allows us how to write powers of complex numbers in polar form. Consider the cubic equation $z^3 = 1$. In the real numbers, this cubic equation has only one solution, i.e. 1. However, if we use complex numbers then the solution changes, there are three solutions for the cubic equation $z^3 = 1$, i.e. [1,0], $[1, 2\pi/3]$, and $[1, -2\pi/3]$. These three cube roots of 1 are referred to as the *cube roots of unity*. Indeed these roots are distinct, as can be seen by representing them in Argand Diagram and regularly spaced round the circle of radius 1, centre the origin. This method can be applied to solve $z^n = 1$ for any positive integer $n$, i.e. to find the $n$th roots of unity. In fact, things are not much different for the equation $z^n = a$, where $a$ is any non-zero complex number. Again the method will give $n$ solutions, and again, these solutions will be regularly spaced round the circle centre the origin. However, the circle may not have radius of 1, and non-real solutions may not occur in conjugate pairs. So, complex numbers can be used to find the solutions of any polynomial equation.

### 1.3 The electrical circuit systems

In the case of EEE, the simplest systems are the electrical circuits contain only resistors. Furthermore, these systems are called *direct current circuits* (DC). It is mathematically defined by any device that obey the relationship $v = iR$, i.e. Ohm's Law, the $R$ is measured in *ohms* whereas $v$ and $i$ are measured in volts and amperes respectively. There are two other components that are commonly found in electrical circuits, i.e. inductors and capacitors. They obey the relationships shown in the Figure 1, where $C$ and $L$ denote the values of those components, in farads and henrys, respectively. The figure also shows the symbol used by EEE when drawing electrical circuit diagrams. Such particular importance systems which consist of resistors, inductors, and capacitors, are called the *alternating current circuit* (AC).

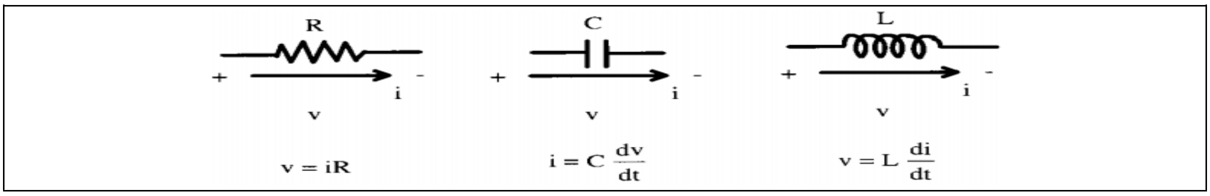



*Figure 1*. The three standard electrical components used in circuits [3]

### 1.4 The abstractions of complex numbers

The AC systems, when the input is a single frequency wave, produce an output at the same frequency which may be phase shifted with a scaled amplitude. Therefore, a single frequency wave can be represented as a *phasor*. Hence, the AC systems can be represented as *complex numbers* which rotate the phasors and change the amplitudes. For example: the AC system is $j$. For any input, the system will produce output with the phasor rotation of $\pi/2$ without changing the amplitude. Many people puzzled on the AC problems (see Figure 2), one of them was Rayleigh (1842 – 1919). However, he eventually had solved his problem at the end of nineteen century using complex exponential [3]. In DC systems, Ohm's Law can be applied, which means if voltage is given then the currents are the proportion of voltage and the resistors. In contrast, it is very difficult to describe the characteristic of the currents in AC systems using both Ohm's and Kirchhoff's Laws. However, when all quantities can be represented as complex numbers, Ohm's law can then be applied in AC systems.

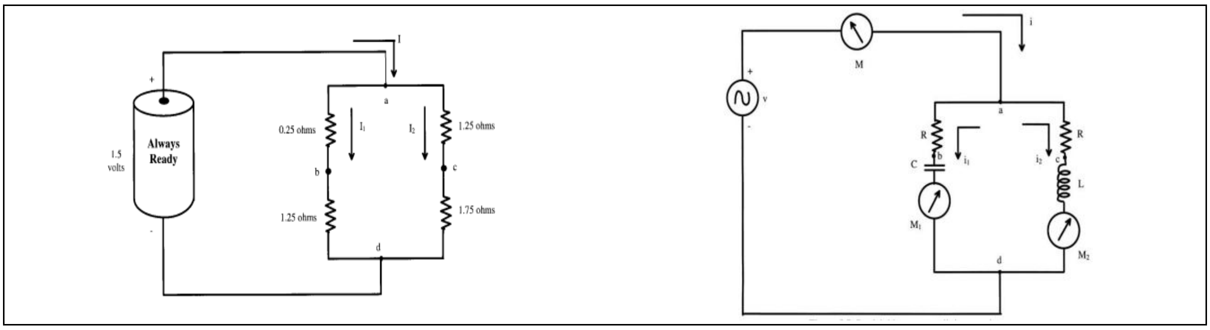
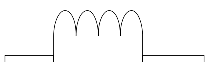
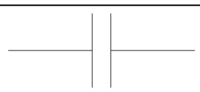



*Figure 2. On the left is a simple DC whereas on the right is the Rayleigh's puzzle AC*

Besides Rayleigh, Charles Proteus Steinmetz (1865 – 1923) wrote his seminal work on solving AC problem using complex numbers [4]. Prior to his work on complex numbers, solving AC problems had to be carried out using calculus, a very challenging and time-consuming processes. Steinmetz was able to reduce the very complex AC theory to his words, "a simple problem in algebra". The key concept in his simplification was the concept of phasors, i.e. the associate complex numbers of the sinusoidal functions. For example $V = 100\cos(\omega t + \pi/2)$ has its associate phasor as $Z = 100\angle\pi/2$. By representing voltages and currents as phasors, Steinmetz were able to define a quantities called *impedances*, i.e. the associate complex numbers of the electrical components (see Table 2). Using Ohm's Law and impedances he then developed the relationship between voltages and currents and their associate phases in one algebraic operation.

*Table 2*. The three standard electrical components used in circuits and their associate impedances [4]

| Electrical components | | Impedances |
|---|---|---|
|  | $R$ | $Z_R = R$ |
|  | $L$ | $Z_L = j\omega L, \omega = 2\pi f$ |
|  | $C$ | $Z_C = -j\frac{1}{\omega C}, \omega = 2\pi f$ |

## 2 THEORETICAL BACKGROUND

In this section the concept formation and the three stages of concept formation [5], i.e. interiorization, condensation, and reification, were discussed. This framework was then used to analyse the difficulties in learning complex numbers for this study.

### 2.1 Concept formation

This subsection discusses the nature of concept formation [5]. Sfard proposed that to operate on mathematical concepts meaningfully students should be able to treat these concepts as objects. Using this argument, I proposed that if a complex number presented, the EEE students should be able to treat this complex number as an object, both algebraically in basic form, polar form or exponential form, and geometrically in Argand diagram, without having to go into the operations that produced this complex number. To arrive at this level of concept formation, it is necessarily to go through three hierarchically ordered stages of concept formation, namely *interiorization, condensation,* and *reification* [5]. Each stage involves working with the related operations and with subsequent mastery of the operations algebraically and geometrically.

### 2.2 The three stages of concept formation

Here, I discusses the three stages of concept formation of complex numbers based on Sfard's theory of concept formation [5]. Firstly, I define *interiorization stage* of complex numbers as the stage when elementary operations on complex numbers were involved, e.g. elementary addition in basic form, elementary multiplication and/or division in polar or exponential form. Secondly, I define *condensation stage* of complex numbers as the stage when further operations on complex numbers were involved, e.g. quotient problems involving addition, multiplication, and division. Finally, I define *reification stage* of complex numbers as the stage when complex numbers were used in de Moivré's Theorem to solve polynomial equation problems and abstracted in AC systems to explain the electrical characteristics.

## 3 METHODOLOGY

Because the aim of this paper was to explore the difficulties faced by EEE students in grasping the concepts, uses, and abstractions of complex numbers, an in-depth analysis including the analysis on the complex numbers tests and EEE students written works, was carry out. These documents were collected from Semarang Polytechnic, Indonesia. These documents were the complex number tests from two lecturers for one hundred and fifty one EEE students. Followed by an in-depth interview with two lecturers. In this paper, I reported the lecturers' responses to two questions: (a) What kinds of representation used while teaching concepts of complex numbers? and (b) To what extent can the concepts, uses, and abstractions of complex numbers be dispensed with? The two lecturers were interviewed on separate occasions. Each interview session was audiotaped and required approximately one hour to complete.

## 4 FINDINGS

This section discusses the in-depth analysis on: (a) the complex numbers test, (b) the EEE students' written works, and (c) the lecturers' responses.

### 4.1 The complex numbers test analysis

The complex numbers test comprised six problems (see Table 3 in Appendix). Table 3 shows the questions and the corresponding objectives and levels of concept formation. Question 1a was a sum of three complex numbers problem. In this case, three complex numbers were given in three different forms, i.e. basic, polar, and exponential forms. Addition in complex numbers can only be done in basic form. Hence the conversions from polar and exponential forms to basic form were required. Question 1b was a quotient of complex numbers problem, where the numerator was a product of two complex numbers and the denominator is a sum of two complex numbers. So to answer question 1b, EEE students should do three steps. Firstly, they should multiply two complex numbers using polar or exponential form. Hence, conversions from basic form to polar or exponential form was required. In this case EEE students could use the concept of Pythagorean Theorem and trigonometric function, i.e. tangent inverse. Secondly, they should add two complex numbers using basic form. Hence, conversions from polar and exponential forms were required. In this case, EEE students could use the concept of trigonometric functions, i.e. sinus and cosine. Finally, they should divide two complex numbers using polar or exponential form. Again, conversions from basic form to both polar and exponential forms were required. Question 2a was another quotient of complex numbers problem. Unlike the question 1b, EEE students could directly answer question 2a using polar or exponential forms. Hence, conversions from basic form to polar or exponential form were required. Question 2b was a conversion problem, i.e. conversion from polar or exponential form to basic form. Question 3a was the reciprocals of a complex number problem where each reciprocal on the right hand side was a quotient of complex numbers. Again the three steps on question 1b were required to answer question 3a. Question 3b was another conversion problem, i.e. from basic form to polar form of the result in 3a. Question 4a and 5a were the uses of complex numbers problems. In the case of question 4a, complex numbers were used to find the square roots of a complex number, whereas question 5a, to find the cube roots. So, to answer question 4a and 5a, de Moivré's Theorem was applied using polar or exponential form. Hence, conversions from basic form to polar or exponential form were required. Question 4b and 5b were another conversion problems, i.e. conversions from polar or exponential form to basic form. Question 6a was abstractions of complex numbers problem. Here, the concept of impedances, i.e. the complex form of the electrical quantities, was used. Furthermore the subtotal and the total impedances were calculated based on their arrangement, i.e. series followed by parallel. Question 2b was another abstractions of complex numbers problem, in this case the concept of phasors was used. Furthermore, Ohm's Law was applied to find the currents.

From the analysis, the concepts of complex numbers are indeed very challenging, e.g. how a particular form is best for particular operation, then how to convert from one form to another, how to use the concepts in de Moivré's Theorem, how to abstract this concept in AC systems, etc. However, none of the question in this test attempted to test EEE students' ability on geometric representation of complex numbers. Historically, only when it was possible to represent complex numbers on the Argand Diagram then it was possible to conceive complex numbers structurally. Without the geometric representation in Argand Diagram, it was possible to operate on the complex numbers but the corresponding understanding of what the operations meant was not forthcoming.

### 4.2 The students' written works analysis

A large variety of works on complex numbers problems were identified and categorised into three major categories which describe as *interiorization*, *condensation*, and *reification* corresponding to the three stages of concept formation [5]. *Interiorization*. These group of students succeed on working the elementary operations on complex numbers. There were three types of works at this level. Only one EEE student, namely Alifa, exhibited good *metacognition skills on complex numbers* because she knew which form best for which operation both algebraically and geometrically and knew when she had the answer. Some EEE students succeed on working these operations algebraically but failed to represent the geometric representation. These works showed that they had *lack of attention* on using geometric representation in Argand Diagram to monitor their works. Some EEE students succeed on working these operations algebraically but using the incorrect selection form. The work of these EEE students showed that they had *lack of attention* on which form best for which work. Furthermore, they also had lack of attention on using geometric representation in Argand Diagram to monitor their works. *Condensation*. These group of students succeed on working the further operations on complex numbers. Similar to the previous level, there were three types of works at this level, i.e. Alifa's, no geometric representation, and incorrect form. *Reification*. These group of students succeed on using complex numbers in de Moivré's Theorem and abstracting complex numbers in AC systems. There were two types of works at this level, i.e. Alifa's and no geometric representation.

Because EEE students were not interviewed, interpretations are only based on analysis of EEE students works. This is one of the limitations of this paper. Those EEE students who solved complex numbers problems successfully demonstrated a sound conceptual knowledge of the selections and conversions, as they kept selecting which form best for which problem and converting from one form to another. In the case of Alifa, she exhibited good metacognition skills on complex numbers because she was able to work on complex numbers problems algebraically and geometrically and knew when she had the answers. In contrast, the rest EEE students seemed had lack of attention on using geometric representation. Because of this, then EEE students tended to ignore the monitoring of their works and the details of their works. Furthermore, no evidence of the use of exponential form of complex numbers. Exponential form is very important, because of its simplicity, the multiplication and division of complex numbers can be done in simpler way. Hence using exponential form tends to produce less errors. Moreover, exponential form allows EEE students to extend their abstractions of complex numbers in AC systems, e.g. explaining the derivative of the AC systems.

### 4.3 The lecturers' responses to the two questions

I interviewed two lecturers from EEE department of Semarang Polytechnic. The objectives of interviews with the lecturers were to find out how the concepts, uses, and abstractions of complex numbers were taught to EEE students. Interviews with the lecturers were conducted to help me understand the keys ideas on which they had believed EEE students had to focus when EEE students work on complex numbers problems. *What kinds of representation used in teaching concepts of complex numbers?* These two lecturers reported that they introduced both algebraic and geometric representations only for basic form. Furthermore, in the worked out examples and practices the geometric representation was less used. *To what extent can the concepts, uses, and abstractions of complex numbers be dispensed with?* The first lecturer, namely Winar, reported that the three ordered hierarchically forms were taught to his students algebraically. Furthermore, he provided the rationale of teaching these form, i.e. how a particular form was best for a particular operation. By this he meant, basic form was best for addition and subtraction, polar and exponential forms were best for multiplication and division. However, he only used polar form when working multiplication and division. Whereas the second lecturer, namely Yani, reported that she only taught the first two forms, i.e. basic and polar forms algebraically. Furthermore, she explained that she only used basic form for all operations of complex numbers. Winar reported that the abstractions of complex numbers were taught to his students algebraically, i.e. concept of phasors and impedances. Again, he provided the rationale of teaching these abstractions, i.e. how complex numbers can be used to solve the AC systems problems. Whereas Yani reported that the uses of complex numbers in de Moivré's Theorem applications were taught to her students algebraically, i.e. to solve polynomial problems.

Because I did not have access to classroom teaching, I cannot comment on the nature of the delivery of the lessons. This is another limitations of this paper. Nevertheless, these two lecturers had a structure on teaching the concepts, uses, and abstractions of complex numbers. These two lecturers understood that the concepts of complex numbers were challenging. However, they had lack of attention on using the geometric representation of complex numbers in Argand Diagram. This made the EEE students did not ensure whether they had the answers. Furthermore, they had lack of awareness on the importance of exponential form of complex numbers problems. E.g. the uses of complex numbers in de Moivré's Theorem applications could be extended for any number, the abstractions of complex numbers using exponential form can be extended for further explanation of the characteristic of the AC systems, e.g. the derivative of the sinusoidal functions. The teaching on complex numbers were only stressing on the algebraic representation and focussing up to polar form could be another reasons why EEE students faced difficulties in learning complex numbers.

### 4.4 Implication for teaching

The visual nature of the concepts, uses, and abstractions of complex numbers means that it is possible for lecturers and EEE students to point to specific levels as they appear the categorization. How can lecturers help EEE students who categorised in the lower stages of concept formation of complex numbers? Lecturers could offer EEE students a set of solutions in the reification stage and compare and contrast these against those that are in the condensation and interiorization. EEE students could be asked to discuss from where those errors came and how they could improve upon their written works. Furthermore, EEE students in the condensation stage could be encouraged to check the conceptual knowledge necessary for the uses and abstractions of complex numbers problems. As the most important help, lecturers should use both the algebraic and geometric representations across the concepts, uses, and abstractions of complex numbers to develop EEE students' metacognition skills on complex numbers. Hence, they would be able to select which form for which operation, explain both the algebraic and geometric meanings, and know when they have the answers.

## 5 SUMMARY

This paper contributes to a piece of work in which the difficulties on learning the concepts, uses, and abstractions of complex numbers were faced by EEE students. The two factors of how challenging these concepts, uses, and abstractions and how lecturers deliver these lessons could be the factors behind these phenomena. However, the time constraint when delivering these lessons could be also considered as another factors. As historically, it took so long for mathematicians to accept the concepts, uses, and abstractions of complex numbers. There is a need to consider the kinds of teaching which supports higher stage of concept formation EEE students on working with complex numbers problems. By extending the concepts, uses, and abstractions on complex numbers may strengthen EEE students' competence on grasping complex numbers.

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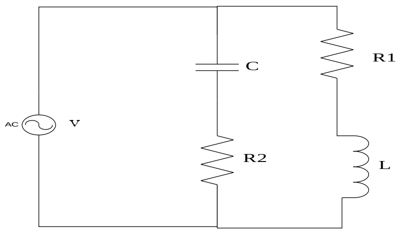
## APPENDIX

Table 3. The questions and the corresponding objectives and level of concept formation

| No. | Questions | Objectives | Level of concept formation |
|---|---|---|---|
| 1. | Given three complex numbers: $z_1 = 10 + j22$, $z_2 = 20\angle 38^0$, $z_3 = 12e^{-j0,23\pi}$ | Given three complex numbers in basic, polar, and exponential forms algebraically. | |
| | a. Find the value of $z_1 + z_2 - z_3$ . | To test EEE students ability to add three complex numbers. | Interiorization |
| | b. Find the value of $\frac{z_1.z_2}{z_1+z_2}$ . | To test EEE students ability to work on a quotient of complex numbers. | Condensation |
| 2. | Given three complex numbers: $z_1 = 2\sqrt{2} + j2\sqrt{2}$, $z_2 = \frac{9}{2} - j\frac{9}{2}\sqrt{3}$, $z_3 = 12\angle 30^0$ | Given three complex numbers in basic and polar forms. | |
| | a. Find the value of $\frac{z_1.z_2}{z_3}$ | To test EEE students ability to work on a quotient of complex numbers | Interiorization |
| | b. Show the result in basic form. | To test EEE students ability to convert complex numbers from polar form to basic form. | Interiorization |
| 3. | Given two complex numbers: $z_1 = \frac{1+j}{3-j2}$, $z_2 = \frac{4+j4}{-12+j12}$ | Given two quotients of complex numbers. | |
| | a. Find the value of $z_3$ if $\frac{1}{z_3} = \frac{1}{z_1} + \frac{1}{z_2}$ | To test EEE students ability to work on reciprocals of complex numbers. | Condensation |
| | b. Show the result in polar form | To test EEE students ability to convert complex numbers from basic form to polar form. | Condensation |
| 4. | Given a complex number: $z = 10 - j5\sqrt{5}$. | Given a complex number in basic form. | |
| | a. Find the value(s) of $\sqrt{z}$ | To test EEE students ability to apply de Moivré's Theorem to find square roots of a complex number. | Reification |
| | b. Show the result in basic form | To test EEE students ability to convert complex numbers from polar form to basic form. | Reification |
| 5. | Given a complex number: $z = 2\sqrt{2} + j2\sqrt{2}$. | Given a complex number in basic form. | |
| | a. Find the value(s) of $z^{\frac{1}{3}}$ | To test EEE students ability to apply de Moivré's Theorem to find cubic roots of a complex number. | Reification |
| | b. Show the result in basic form | To test EEE students ability to convert complex numbers from polar form to basic form. | Reification |
| 6. | Given an electrical circuit below: | Given an electrical diagram pictorially. | |

| | | | |
|---|---|---|---|
| |  | | |
| | Given the electrical quantities below:<br>$V = 100\angle 0^0$ volt<br>$f$ = 50$H$<br>$C_1$ = 200$\mu F$<br>$R_1$ = 30$\Omega$<br>$L_2$ = 340$mH$<br>$R_2$ = 35$\Omega$ | Given the electrical quantities algebraically. | |
| | a. Find the value of the total impedance | To test EEE students ability to abstract the concepts of complex numbers in AC system to find the total impedance | Reification |
| | b. Find the value of the current in each branch. | To test EEE students ability to abstract the concepts of complex numbers in AC system to find the currents | Reification |