

Mathematical beauty in service of deep approach to learning

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INTRODUCTION

In the fall of 2014 I taught ‘02601 Introduction to Numerical Algorithms’ to a class of 86 engineering students at Technical University of Denmark. The course employed basic calculus and linear algebra to elucidate and analyse canonical algorithms of scientific computing. A major part of the course was hands-on MATLAB programming, where the algorithms were tested and applied to solve physical model-based problems. To encourage a deep approach, and discourage a surface approach to learning, I introduced into the lectures a basic but rigorous mathematical treatment of crucial theoretical points, emphasising the beauty of the underlying mathematical structure. Into this I integrated frequent and activating dialogue with the students. In section 1 I describe the course and the students in more detail. Section 2 details and justifies the pedagogical elements I introduced into the lectures; my central hypothesis is also given there. The results of the experiment are presented and discussed in section 3.

1 THE COURSE AND THE STUDENTS

The curriculum of the 5-ECTS point course consisted of two major components, one concerned with the theoretical foundation, the other with the practical aspects and use of numerical algorithms. This duality was reflected in the teaching and learning (T&L) material and activities, and, to a lesser extent, in the assessment. During classical, large-auditorium lectures, calculus and linear algebra were used to explain the operation and analyse the performance of numerical algorithms for root finding, optimisation, data fitting etc. After each lecture, the students worked in groups of 2–3, where they primarily used the algorithms and MATLAB to solve practical numerical problems. Based on this work, each group delivered 3 extensive written reports, and the average grade for these reports constituted the basis for the final assessment. At the end of the semester the students were assessed individually via 10–15 min. oral exams probing primarily the theoretical knowledge and the actual involvement of the students in their group’s work. The impact of the performance at the oral exam was limited to small changes in the report grade. The course used the textbook of Chapra [1], which includes both a mathematical and a MATLAB-oriented practical approach

to the subject, but where the focus is clearly on the applied aspects. I here argue that, similar to the divide in the nature of the curriculum, of the T&L and of the assessment, the students themselves fell into two categories – those with primary interest in practical programming and numerical problem solving, and a smaller group with a strong interest in a mathematical approach to numerical algorithms. Indeed, about 33% of the students were from BEng study programmes¹. Previous lecturers told me that BEng students often had fewer mathematical prerequisites and achieved lower grades in the course compared to their BSc and MSc counterparts. To test this student model, at the beginning of the first lecture I administered a 3-minute informal and anonymous quiz to the students. The quiz gauged the students' expectations for the course and the degree to which they fulfilled its prerequisites. The questions/tasks were: A) what do you primarily expect to learn about in the course – computers, MATLAB and programming, or the mathematical treatment of algorithms (convergence analysis, error estimates etc.)?; 1) write the Taylor series for the function $f(x) = e^x - 1$ about the point $x_0 = 0$; 2) write a MATLAB command plotting the graph of $\ln(x)/x$ at 100 points in the interval $x \in [2, 5]$; 3) write a formula that expresses Newton's II law; 4) what is the limit of $\ln(x)/x$ as $x \rightarrow \infty$? I scored the tasks 1–4 on a scale from 0 (no or completely wrong answer) to 1 (fully correct answer). Fig. 1 shows quiz results with 95% confidence intervals (in Fig 1a computed for 48 answered sets from a student population of 86, in Fig 1b for the 48 sets).

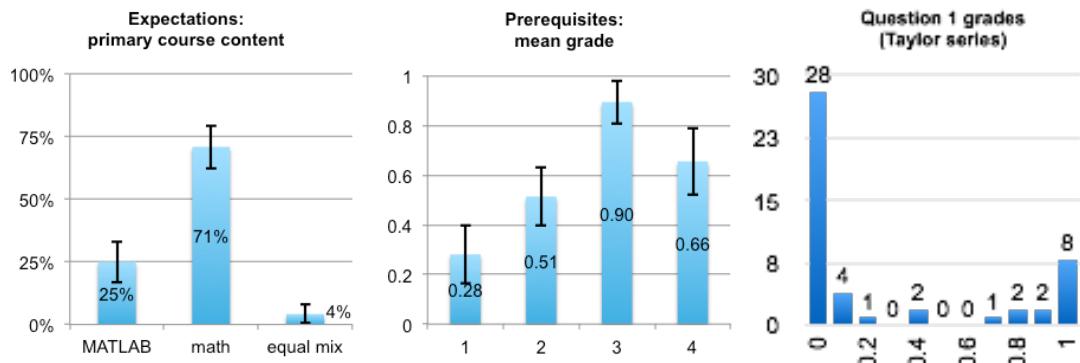


Fig. 1. a) expectations, b) prerequisites, c) task 1 grades.

The mean grade for the Taylor series, a fundamental and crucial prerequisite of the course, was less than 0.4. The percentage of satisfactory and unsatisfactory *delivered* solutions (grade above 0.7 and below 0.3, respectively) was 25% and about 69%, respectively. The gap clearly visible in Fig. 1c supports my assumption of a divide in the students' prerequisites. What about their interests? Only math as the primary focus was expected by about 82% of those who wrote a correct Taylor series, and by about 68% of those who did not, consistent with my assumption of there being a 'mathematically skilled and interested' group and a 'less mathematically skilled and more MATLAB-oriented' group. At least 60% expected a mathematical focus in the course, well-aligned with my intentions. The mean grade for task 4 was up to 0.8, but the answer ('zero') was easy to communicate and copy, in contrast to the more complicated Taylor formula (I did witness some students sharing specifically answer no. 4 during the quiz). I stress that *I never used the student model presented here apologetically* ('blame the student', level 1 of thinking about teaching [2, pp. 17–18]), but rather to try to set an appropriate stage for quality learning for as many students as possible – by reinforcing prerequisite mathematical tools and concepts on-the-go in the lectures.

¹ BEng, BSc and MSc are Bachelor and Master engineering study programmes at Technical University of Denmark, see [4] for a description.

2 NEW ELEMENTS AND THE CENTRAL HYPOTHESIS

Biggs and Tang [2, pp. 24–29] define and discuss ‘surface’ and ‘deep’ approaches to learning. A deep approach demonstrably results in better learning, involves all cognitive levels of learning activities (from memorisation to hypothesising and reflection), and thus a good teaching practice should encourage a deep approach and discourage a surface approach. Students who are not at the outset intrinsically motivated [3] can become so by *social motivation* (inspiration by an enthusiastic teacher) and by building up a solid *knowledge base*, leading to a sense of *ownership of their learning* [2, pp. 34–39]. I here add that many students who are already intrinsically mathematically motivated, in my experience, respond well to rigorous, in-depth approach to the subject. A mathematically honest and complete treatment carries an inherent satisfaction and sense of ownership, and the more challenging aspects may trigger achievement motivation, as defined in [2, p. 35]. Conversely, I argue that a perfunctory, shallow presentation of the underlying math, where real proofs and solid justification give way to *at most* using ‘intuitive’ arguments and illustrative isolated special cases, directly encourages a surface approach to learning in *all* students. If crucial arguments behind the algorithms are not properly understood, the students may suffer ‘undue anxiety and low expectations of success,’ as put in [2, p. 26]. Apparent lack of structure and insufficient challenge may be discouraging even to intrinsically motivated students. The curriculum of 02601 encompassed both the mathematical basis and practical applications of numerical algorithms, and so in principle it set the stage for a well-motivated and well-rounded presentation of the subject, encouraging a deep approach to learning. However, in my opinion, the course material and the assessment format actually somewhat undermined this, as follows. The highest-level stated learning objectives for 02601 included the action verbs (Bloom’s revised taxonomy [2, Table 7.2, p. 124]) ‘analyse’, ‘derive’, ‘critique’ and ‘evaluate’. I found the textbook [1] to not deliver enough theoretically to be well-aligned with the above learning objectives. On several occasions, for example in [1, Section 7.2.1, p. 187] regarding the Golden-Section Search (GSS), the textbook treated the topic non-rigorously. The GSS is an optimisation algorithm capable of finding minima of strictly unimodal, univariate, real-valued functions. It is a special case of the trisection method, characterised by a particularly efficient choice of the next sampling point x_3 given the previous sampling points x_1 , x_2 and the values $f(x_1)$, $f(x_2)$, see Fig. 2a.

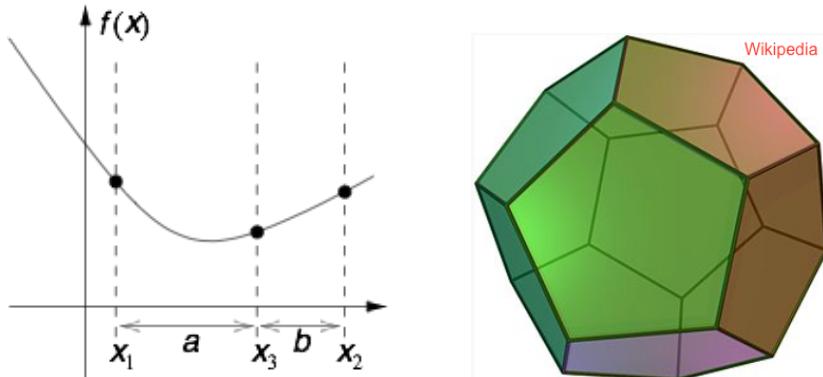


Fig. 2. What does efficient optimisation have to do with Platonic solids?

It turns out that the special choice of a and b satisfying $(a+b)/a = a/b$ makes the optimisation significantly faster than general trisection, in that one of the ‘old’ sampling points x_1 , x_2 and the pertaining function evaluation $f(x_1)$, $f(x_2)$ can be reused in the next iteration. But in this case $a/b = \phi = (1+5^{1/2})/2 \approx 1.62$ is the *golden ratio*, an irrational number studied at least since Euclid [5] and occurring frequently in

geometry (for example in relations between Platonic solids) and, e.g., in connection with Fibonacci sequence. There are also numerous alleged uses of ϕ in music, art and architecture. A proof of the optimality of ϕ as a parameter in the trisection method, well within the mathematical reach of the students, was omitted in [1], and in my view this amounted to passing an excellent opportunity to deepen the treatment and include exciting and diverse examples of application. Next, the written reports – the basis for the summative assessment in the course – were graded at the group level and dominated by the practicalities of MATLAB programming. Some of the exercises could be done without a deep understanding of the employed algorithms. Thus, students not intrinsically motivated in the underlying mathematics would perhaps be moved to take a surface approach to learning and hence miss crucial points regarding the applicability and performance of numerical methods in relation to posed problems, regardless of their MATLAB programmes running well. Coming into an established course, I was not at liberty to change the teaching material, the teaching form and the assessment. I had to innovate within set boundaries. This constricted me mostly to level 2, and just a portion of level 3 of teaching according to [2, pp. 16–20]. I introduced into my lectures a proper mathematical treatment of the key theoretical points. I derived proofs and justified mathematical statements on the blackboard, in collaboration with the students whenever feasible. I tried to take my time with the derivations, constantly described what I was writing down, frequently asked the audience what I should write next, encouraged intermediate questions, regularly engaged individual students in direct dialogue, and frequently revisited the motivation and reasoning behind what was going on. I enthusiastically strived to showcase the beauty of the underlying mathematical structure, such as the above mentioned unexpected connecting role of ϕ , or the justification of Richardson extrapolation in adaptive numerical quadrature, to intrigue the students and motivate them to follow the derivations. This often allowed me to lecture at a higher conceptual level and engage and activate the students in a way that was organically integrated into the lectures. I also tried to unify the reasoning behind different areas of application of numerical algorithms (e.g. in solving initial value problems, in data fitting and in numerical integration: we either do not know or cannot handle analytically the exact function behind the solution, but appropriate mathematical reasoning can still give us useful numerical results!). I stress that the extra math was just an additional element in the teaching, and did not at all take up the majority of the time. Rather, it consisted of several sequences/derivations at carefully selected, crucial points in the curriculum. It was *not* an elitist measure – it was, for example, clearly stated that the assessment was criterion-referenced – and my hope was that students from both ‘groups’ of Section 1 would benefit. **My central hypothesis was:** *a basic but rigorous mathematical treatment of numerical algorithms, in excess of what was shown in the textbook [1], would encourage a deep approach and discourage a surface approach to learning in the students, even in the large-classroom setting.*

3 RESULTS AND DISCUSSION

What effect did the introduced elements actually have on the students’ learning? Here are the relevant results of several different evaluations done during and after the lecturing period.

About one-half into the course I interviewed 3 separate group-work classrooms of students about their impressions of the teaching and learning so far. All 3 classrooms said the teacher-student dialogue/contact was very good, and there was no need to

increase or decrease its amount. In two of the classrooms there were students who said they disliked being singled out for questions during lectures, because they did not know all the answers. The students, however, were very positive about the dialogue where the learning objectives were recapitulated after the lecture. Two of the classrooms strongly agreed, and one agreed, that the course was overall good. One student commented that there should be more theoretical questions in the compulsory written reports. My overall impression from this mid-term evaluation was that some students were still adjusting to the increased amount of dialogue in the large-classroom lectures, but the overall sentiment was positive. In two of the classrooms there were students who said they did not know what to expect at the oral exam. In each lecture I explicitly stated the learning objectives at the beginning, and then recapitulated them in a direct dialogue with the students at the end. The intended learning outcomes of the course were also explicitly stated. However, I do understand that the additional math in the lectures may still have confused some students about exactly how much of what was presented they were expected to perform at the oral exam.

I am currently enrolled in LearningLab DTU's Teacher Training Programme². In a peer-coaching arrangement within the programme, one of my lectures was attended by three assistant professors undergoing the same teaching and learning course at Technical University of Denmark as I. The peers assessed my teaching, the students' reactions and engagement, and the general atmosphere during the lecture. In summary, the positive portion of the written feedback included that: I connected well with the students, the size of the class taken into account; I activated the students and kept alert those 'hiding' in the back; the calculations on the blackboard slowed the pace of the lecture and allowed the students to participate in the process/reasoning step-by-step, which also resulted in good dialogue; I was very secure in my subject matter; the lecture went really well; the interaction via questions to specific, directly addressed students worked surprisingly well. The peers suggested the following improvements: shorter introduction, engaging the students at an earlier stage in the lecture, perhaps in finding the relevant questions; a more hands-on example (like a data set) to demonstrate first-hand how to process it and hence introduce math 'through the back door'; part of the students did benefit from the derivation of the normal equations in linear regression, but some quickly found other activities during the 20-minute session, and perhaps lost the concentration for the last part of the lecture—think about how much time is reasonable to put aside for the extra math, and put this at the end of the sequence, so fewer students lose out on the remainder of the lecture. Hence, the extra math did help clarify the subject matter, improve my teaching style and foster activating dialogue with the students. The comment about being secure in the subject matter illustrates my previous point in action: I argue that taking a detailed approach to the underlying mathematics, not merely stating facts but also proving them, 'leaving no stone unturned,' demystifies the subject matter, instills additional confidence in the students about their lecturer and themselves, and encourages a deep approach to learning at least through social motivation. The price I paid was losing a part of the audience in the process; leaving the deep derivations to the end of the lecture might help.

At the end of the semester the students made a standardised online evaluation of the course content, teaching and assessment. Here are some relevant results:

² <http://www.learninglab.dtu.dk/english/kurser/undervisere/udtu>

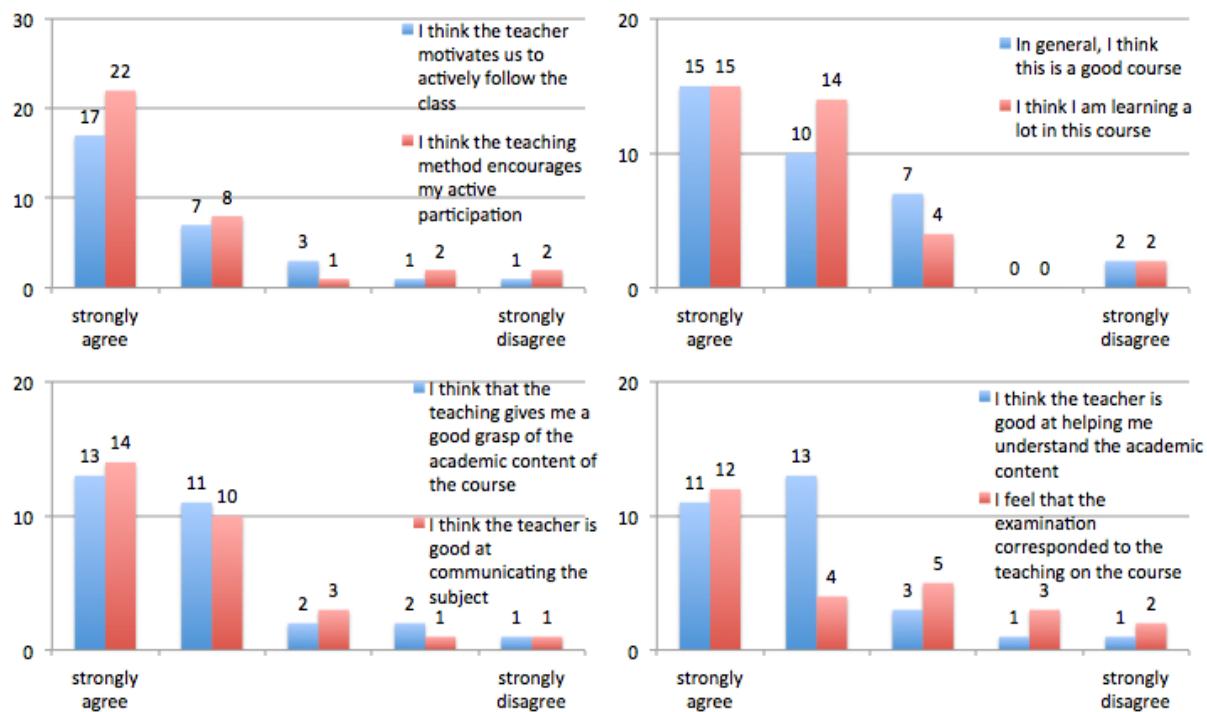


Fig. 3. Student evaluation of the course.

This overall supports my central hypothesis. Also, 76.5% of the enrolled students achieved the average grade (7) or better, namely 23.5%, 27.1% and 25.9% with grades 7, 10 and 12 respectively (C, B and A, respectively, on the ECTS scale). In their written comments the students described the teaching "good", "very good", "pedagogical" and "systematic." Below I focus on the comments that proposed changes. One student wrote that the lectures were good, but that there should be a little less focus on the derivations, since these were very difficult to follow. The same student added that they enjoyed the real-life cases and examples I employed. Several other students also mentioned these examples in a positive light (at least one wrote outright that the examples were motivating). These comments are consistent with a suggestion I got at peer coaching (see above), and point to how I can make the additional math more 'digestible' in the future: by adding more physical/model-based examples leading to, or perhaps intermixed with the derivations. A student wrote that it was motivating I posed questions to the auditorium and expected answers. Another wrote that I was very engaged in the subject matter, and that my questions to the auditorium worked very well for them. That student added that the derivations on the blackboard could be a little too detailed at times. Another comment suggested it was sometimes impossible to see what was going on at the blackboard, and that the argumentation for the derivations was sometimes incomplete. The intelligibility of some of the derivations/arguments might have suffered because of the sheer size of the auditorium, in spite of me always asking whether what was shown was understood and satisfactory. I got two comments saying that the tempo of the lectures was sometimes too high. The extra derivations did put a strain on my timing on some occasions, and this might have forced me to pick up the pace in other parts of the lecture. I shall certainly consider this time/depth trade-off more in the future. While there were many positive assessments of the exam, several students wrote that the oral exam was at the (high) level of the lectures, but unexpectedly much more theoretical and at a higher level than the written reports. At least two students suggested independently that there be more high-level theoretical questions in the reports in order to better prepare for the oral exam. This misalignment between the lectures, the written reports and

the oral exam needs to be corrected, perhaps by providing more formative feedback in connection with the reports (also a request made by some of the students). Several students wrote the connection between the lectures and the group exercises was very good. A recurring criticism was the large workload, especially in connection with the written reports. Combined with the fact that the vast majority of the exercises in the written reports were MATLAB-oriented, this may have lessened the enthusiasm for the extra math in some of the students. Large work load promotes a surface approach to learning [2, p.2 6], and this issue will have to be corrected in the course. At least three comments said the course was difficult for BEng students or generally those with inadequate mathematical prerequisites. One comment explicitly pointed to the differences in the mathematical prerequisites of students of BSc Mathematics and Technology line against the rest. This supports my student model of Section 1, but it also indicates I should do more to make the course accessible to a broad range of students – although *Fig. 4a* suggests the theoretical level and the workload were not, on average, deemed too high.

At the end of the course 23 students completed a “Course Experience Questionnaire” (CEQ) [6] aimed at an assessment of the student experience of the course and of my teaching. The students graded 22 statements on a scale from 1 ('fully disagree') to 5 ('fully agree'), and I here present 10 statements that I find most relevant to my central hypothesis: 1) the course was intellectually stimulating; 2) it seems the curriculum tried to cover too many subjects; 3) I found the course motivating; 4) the course has sharpened my analytical skills; 5) the course has made me more secure in tackling new and unknown problems; 6) the course has made me want to learn more; 7) the course has awakened an interest in the field of the subject in me; 8) the course has developed my problem-solving abilities; 9) the lecturer really made an effort to make the course content interesting to me as a student; 10) the sheer amount of work I had to do in the course meant that I could not learn everything thoroughly. *Fig. 4b* shows the average score for the 10 statements, with 95% confidence intervals for 23 respondents.

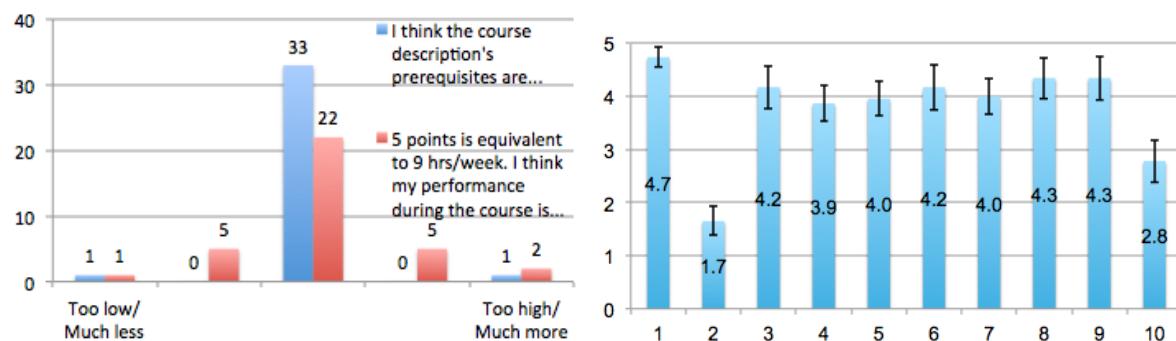


Fig. 4. a) prerequisites and work load, b) Course Experience Questionnaire scores.

I find these results in support of my central hypothesis. The additional math may however have contributed to a perceived increase in the amount of required work, and hence to the rather high score for question 10 and the score of up to 2 for question 2. Several students expressed gratitude for an interesting course in connection with their CEQ answers.

My focus on the mathematical treatment of the subject demonstrably encouraged some students to learn better within the curriculum, and in certain cases to also explore beyond it. While discussing an error estimate for the Simpson's rule in numerical quadrature, I mentioned the curious fact that the rule performs 'better than it should,' in that it integrates third-degree polynomials exactly although the involved

interpolation is only quadratic. Helping me emphasise this peculiarity were the explicit derivations of error estimates for simpler numerical quadrature rules that I made earlier in the lecture. This awoke interest in a student, who approached me after the lecture and asked for a proof of the Simpson error estimate (although this was outside curriculum). I e-mailed a copy of the proof to the student subsequently. The student later achieved the very good grade 10 on the 12-scale. Next, I saw several instances of written reports with nice explanations of, e.g., the significance of the golden ratio in GSS, and of other additional mathematical elements I introduced in the lectures.

Finally, here is what I can only report but not formally document: during and after my lectures, and in the group exercises, I was asked many conceptual, deeper math-related questions, which indicated a deep approach [2, p. 26]. In my opinion, some of the students clearly showed inspiration by the extra mathematical material at the lectures, were eager to ask and answer, to propose solutions and at least in one case also offer comments that related the matter with what they learned in other courses. At the oral exam some students recounted the deeper arguments and explanations of the inner workings of the algorithms. However, I there also saw students who still thought they would only be asked about isolated facts pertaining to the written reports, and were surprised to find that they also had to explain concepts and some math behind the methods.

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